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SOME TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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Successive application of a series of conformal mappings is used to determine the steady-state heat flow and temperature fields in two-dimensional rectangular configurations with mixed boundary conditions. A uniform medium is considered. The analytical solutions are compared with results obtained by means of an electro-hydrodynamic analog.

In many heat-transfer processes there arises the problem of determining the steady-state heat flow and temperature field in two-dimensional rectangular configurations*, in which a maximum and a minimum temperature are specified on two segments of the boundary, respectively, and the rest of the boundary is insulated (Fig. 1).

From the practical point of view, problems of this type, including those with more complex contours, can be solved in an easy and simple way (in the numerical sense) by means of an electro-hydrodynamic analog [1].

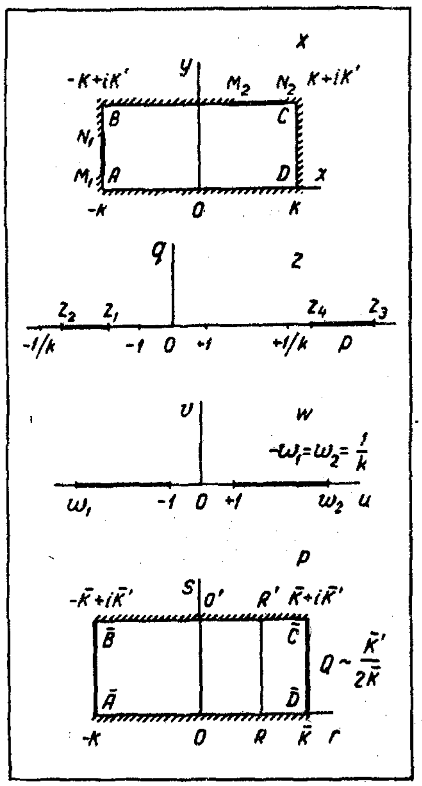

Fig. 1. Scheme of successive conformal mappings.

However, in those cases in which the equipment necessary for the construction of the electrical circuit is not available, one must solve the problem analytically.

In the following we shall demonstrate a simple and easy sequence of operations which, with some practice, provides a rapid solution to the problem before us.

The solution is based on the application of a series of successive conformal mappings, which make it possible to reduce the problem to the simple problem of a rectangular plate, which, with constant maximum and minimum temperatures, is solved by elementary methods.

As is well known [2-4], when an appropriate transformation is applied to the boundary conditions (during the mapping from the original to the image plane) the heat fluxes in the original and image planes are identical, and conformal mappings of the isotherms of the original plane are isotherms of the image. Clearly, in the simple case considered here the temperatures of the diathermic segments of the original and the image should be identical.

Consider the problem of determining the heat flux in a rectangle ABCD in which the segment M_1N_1 of the boundary is held at the temperature T_1 and the segment M_2N_2 is held at T_2 ($T_1 > T_2$). The rest of the contour is insulated (Fig. 1).

To solve this problem, let us map the rectangle into the upper half-plane Z. Further, let us map the upper half-plane Z into the upper half-plane W in such a way that the images of the segments M_1N_1 and M_2N_2 become symmetric with respect to the origin. Finally, let us map the upper half-plane W into the rectangle $\overline{ABCD}$ of the P plane. This final image is solvable analytically, and its heat flux and isotherms can be calculated.

It is well known [5, 6] that the elliptic integral of the first kind

$$u = \int_0^z [(1-t^2)(1-k^2t^2)]^{-1/2}\,dt = \int_0^{\varphi} (1-k^2\sin^2\varphi)^{-1/2}\,d\varphi \tag{1}$$

with modulus $k \leq 1.0$ defines a conformal mapping of the upper half-plane into a rectangle with the vertices K, $K+iK'$, $-K$, $-K+iK'$, where

*Throughout this work the heat flux is to be understood as the flux per 1 linear meter of a bar in the direction normal to the plane of the drawing.

$$K = \int_0^1 [(1-t^2)(1-k^2t^2)]^{-1/2}\,dt = \int_0^{\pi/2} (1-k^2\sin^2\varphi)^{-1/2}\,d\varphi, \tag{2}$$

$$K' = \int_0^1 [(1-t^2)(1-k'^2t^2)]^{-1/2}\,dt = \int_0^{\pi/2} (1-k'^2\sin^2\varphi)^{-1/2}\,d\varphi, \tag{3}$$

and $k' = \sqrt{k^2-1}$ is the complementary modulus. The point 0 of the original half-plane is mapped into the point 0 of the image plane, and the points ±1 and ±1/k are mapped into the vertices of the rectangle.

The conformal mapping of the rectangle ABCD into the half-plane Z is defined by the inverse of the elliptic integral of the first kind, i. e., by the so-called Jacobi's elliptic function, or elliptic sine sn u.

Thus, an arbitrary point x + iy of the rectangle in the X plane is mapped into the corresponding point p + iq in the Z plane by the relation

$$z = p + iq = \mathrm{sn}(x+iy) =$$

$$= \frac{\mathrm{sn}(x,\ k)\,dn(y,\ k') + i\,\mathrm{cn}(x,\ k)\,dn(x,\ k)\,\mathrm{sn}(y,\ k')\,\mathrm{cn}(y,\ k')}{\mathrm{cn}^2(y,\ k') + k^2\,\mathrm{sn}^2(y,\ k')\,\mathrm{sn}^2(x,\ k)}. \tag{4}$$

The elliptic functions cn u and dn u are related to the elliptic sine sn u by the equalities

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \quad dn^2 u + k^2\,\mathrm{sn}^2 u = 1. \tag{5}$$

Consider the mapping of a rectangle in the X plane, with sides a and l, into the half-plane Z. According to the mapping of a half-plane into a rectangle, the vertices of the rectangle will be the points K, K + iK', −K + iK', and −K. Hence a = K', l = 2K, and K'/K = $2a/l$ = m. Now let us determine the modulus k and the complementary modulus k' of the elliptic integral of the conformal mapping. Knowing the value m = K'/K, we find in a table of the elliptic integrals of the first kind [7] with $\varphi = \pi/2$ such values of K' and K whose ratio is equal to m, and we find the corresponding values k' = sin α' and k = sin α. Further, we find the images of the sections M_1N_1 and M_2N_2 in the Z plane, taking into account that all points lying on the sides of the rectangle are mapped into the real axis in the Z plane. Using the equation

$$z = p = \frac{\mathrm{sn}(x,\ k)\,dn(y,\ k')}{\mathrm{cn}^2(y,\ k') + k^2\,\mathrm{sn}^2(x,\ k)\,\mathrm{sn}^2(y,\ k')}, \tag{6}$$

we obtain an expression for the coordinate p in those cases when the segments M_1N_1 and M_2N_2 lie on the sides AB and CD of the rectangle in the X plane, or when these segments lie on the sides BC and CD.

Taking into account that sn(K, k) = sn(K', k') = 1.0, sn(x, k) = sin φ, sn(y, k') = sin φ', one can obtain, after some simple transformations, the expression $p = (1 - k'^2\sin^2\varphi')^{-1/2}$ for the first case and p = 1/k sin φ for the second case.

The values φ and φ' are determined by means of tables of elliptic integrals according to the known relations between the segments M_1N_1 and M_2N_2 and the sides of the rectangle, equal, as stated above, to K' and 2K.

Further, as mentioned above, we must map the upper half-plane Z into the upper half-plane W in such a way that the segments z_1z_2 and z_3z_4 are equal and symmetric with respect to the origin (Fig. 1). This mapping is obtained by means of the bilinear transformation

$$w = u + iv = (Az+B)/(z+C). \tag{7}$$

The three parameters of the transformation (A, B, and C) are determined by the three conditions:

1. The mapping of the point z_1 of the Z plane into the point −1 of the W plane:

$$(Az_1+B)/(z_1+C) = -1.0. \tag{8}$$

2. The mapping of the point z_4 of the Z plane into the point +1 of the W plane:

$$(Az_4+B)/(z_4+C) = 1.0. \tag{9}$$

3. The equality of the segments z_1z_2 and z_3z_4, which requires that the coordinates of the points $-w_1$ and w_2 in the

W plane be equal:

$$-(Az_2+B)/(z_2+C)=(Az_3+B)/(z_3+C). \tag{10}$$

In this way we obtain three equations with three unknowns:

$$\begin{gathered}Az_1+B+C=-z_1,\\ Az_4+B-C=z_4,\\ 2Az_2z_3+B(z_2+z_3)+AC(z_2+z_3)+2BC=0.\end{gathered} \tag{11}$$

This system reduces to a quadratic equation for A

$$A^2-2pA+1=0, \tag{12}$$

where

$$p=\frac{-2z_2z_3+(z_2+z_3)(z_1+z_4)-(z_1+z_4)^2/2-(z_4-z_1)^2/2}{(z_4-z_1)(z_2+z_3-z_1-z_4)}, \tag{13}$$

and consequently

$$A=p+\sqrt{p^2-1}, \tag{14}$$

$$B=\frac{1}{2}(z_4-z_1)-\frac{1}{2}(z_1+z_4)A, \tag{15}$$

$$C=-\frac{1}{2}(z_1+z_4)+\frac{1}{2}(z_4-z_1)A. \tag{16}$$

It should be noted that we choose that root of the quadratic equation (12) which gives a mapping of the points $z > 1$ in the Z plane into the points $w > 1$ in the W plane. The other root gives a mapping of the points $z > 1$ into the points $w < 1$. It can be easily shown that the use of the latter root yields the same results as the use of the first one.

Thus, the bilinear transformation with the coefficients defined by (14), (15), and (16) is a single-valued mapping of the upper half-plane Z into the upper half-plane W, and the images of the segments M_1N_1 and M_2N_2 in the original plane X, in which we are interested, are equal and symmetric with respect to the origin of the W plane.

Now, mapping the half-plane into a rectangle in the P plane, subject to the condition that the modulus $\overline{k} = 1/w_2$, we obtain in the P plane the rectangle $\overline{ABCD}$ with the vertices $\overline{K}$, $\overline{K} + i\overline{K}'$, $-\overline{K} + i\overline{K}'$, $-\overline{K}$ (Fig. 1) where

$$\overline{K}=\int_0^{\pi/2}(1-\overline{k}^2\sin^2\varphi)^{-1/2}\,d\varphi,\quad \overline{K}'=\int_0^{\pi/2}(1-\overline{k}'^2\sin^2\varphi)^{-1/2}\,d\varphi.$$

Clearly, the heat flux from the wall $\overline{AB}$ to the wall $\overline{CD}$ in this rectangle is

$$Q=\frac{\lambda}{2\overline{K}}\overline{K}'\Delta T. \tag{17}$$

It is also clear that the isotherms of this rectangle are straight lines parallel to $\overline{AB}$ and $\overline{CD}$, e.g., RR'. Now, to construct the isotherms in the original plane X we must map the straight lines RR' consecutively into the planes W, Z, and X.

As stated above, the mapping of a rectangle into a half-plane is defined by the elliptic sine sn u,

$$w=u+iv=\operatorname{sn}(r+is),$$

where the real part of the coordinate w is

$$u=\frac{\operatorname{sn}(r,\overline{k})\,dn(s,\overline{k}')}{\operatorname{cn}^2(s,\overline{k}')+\overline{k}^2\operatorname{sn}^2(r,\overline{k})\operatorname{sn}^2(s,\overline{k}')} \tag{18}$$

and the imaginary part is

$$v = \frac{\mathrm{cn}(r, \bar{k})\, dn(r, \bar{k})\, \mathrm{sn}(s, \bar{k}')\, \mathrm{cn}(s, \bar{k}')}{\mathrm{cn}^2(s, \bar{k}') + \bar{k}^2 \mathrm{sn}^2(r, \bar{k})\, \mathrm{sn}^2(s, \bar{k}')}. \tag{19}$$

For the central isotherm 00' (r = 0) u = 0 and w = iv = sn is = i sn (s, $\bar{k}$')/cn (s, $\bar{k}$'), i. e., the imaginary axis of the P plane is mapped into the imaginary axis of the W plane.

The inverse bilinear mapping of the half-plane W into the half-plane Z is

$$z = p + iq = \frac{B - C(u + iv)}{(u + iv) - A} \tag{20}$$

or, after some simple transformations

$$p = [-C(u^2 + v^2) + (AC + B)u - AB]/[(u - A)^2 + v^2], \tag{21}$$

$$q = (AC - B)\, v/[(u - A)^2 + v^2]. \tag{22}$$

The mapping of the points z = p + iq of the Z plane into the original plane X is defined by the elliptic integral

$$x + iy = \int_0^{\varphi} (1 - k^2 \sin^2 \varphi)^{-1/2}\, d\varphi,$$

where

$$\varphi = \arcsin(p + iq) = \arcsin z.$$

However, these formulas are somewhat difficult to use and it is easier to use the inverse transformation

$$z = p + iq = \mathrm{sn}(x + iy)$$

where

$$p = \frac{\mathrm{sn}(x, k)\, dn(y, k')}{\mathrm{cn}^2(y, k') + k^2 \mathrm{sn}^2(x, k)\, \mathrm{sn}^2(y, k')}, \tag{23}$$

$$q = \frac{\mathrm{cn}(x, k)\, dn(x, k)\, \mathrm{sn}(y, k')\, \mathrm{cn}(y, k')}{\mathrm{cn}^2(y, k') + k^2 \mathrm{sn}^2(x, k)\, \mathrm{sn}^2(y, k')}. \tag{24}$$

The system of equations (23)-(24) can be solved by a graphical method, which consists, essentially, in the following: The values p and q are known, as we have already obtained the points z = p + iq of the images of the isotherms in the Z plane. Using different values of y with known values of k and k', we solve one of the two equations, e. g., (24), for x. Substituting these values of x and y into the right-hand side of (23), we seek that pair of values of x and y for which the right-hand side of (23) becomes equal to p.

We have carried out such calculations for several configurations of the segments M_1N_1 and M_2N_2 with respect to the sides of the rectangle (Fig. 2):

1. A rectangle with segments M_1N_1 and M_2N_2 lying on opposite sides and equal to $a/2$. The calculations were carried out for $l = 2a/3$, $l = a$, and $l = 2a$.

2. A rectangle with segments M_1N_1 and M_2N_2 lying on the same side and equal to $a/4$. The calculations were carried out for the same values of l: $l = 2a/3$, $l = a$, and $l = 2a$.

In the first case the heat flux Q was scaled with respect to the heat flux Q_{pl} between the opposite sides of a rectangular plate with sides l and $a/2$, i. e., with respect to the value M_1N_1.

Furthermore, in the case of a square ($l = a$) with the segments on opposite sides we have also constructed the isotherms (Fig. 3).

In the first set of calculations the heat flux varies from an infinitely large value for $l = 0$, when the hot and cold segments lie on the same side of the rectangle and the thermal resistance vanishes, to zero when $l \to \infty$, in which case the thermal resistance is infinitely large.

In the second set of calculations the heat flux varies from 0 for $l = 0$ (there is no heat transfer between two segments separated by a thermal insulation) to a constant value, which is reached, practically, at $l = a$; an increase of the length of the rectangle beyond this value has no effect on the heat-transfer process.

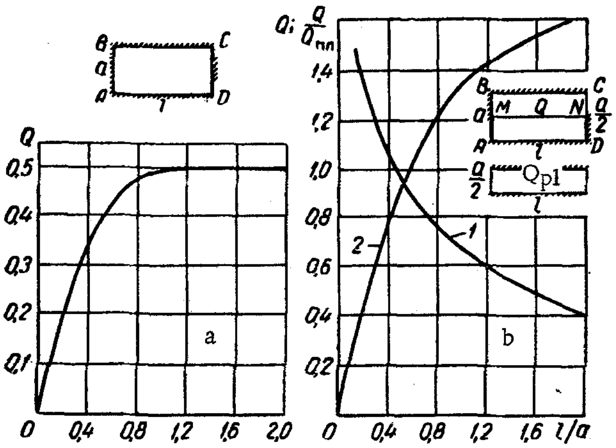

Fig. 2. Heat flux ($1 - Q$; $2 - Q/Q_{pl}$) as a function of the dimensions of the rectangle. a) case 2; b) case 1.

The solution demonstrated above is exact. Thus the analytic results can be used to check the results obtained by means of an analog, and the analog can then be used to obtain certain supplementary results which are difficult to obtain analytically.

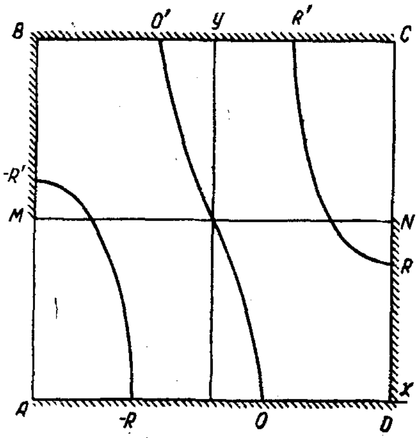

Fig. 3. Isotherms constructed by the method of successive conformal mappings.

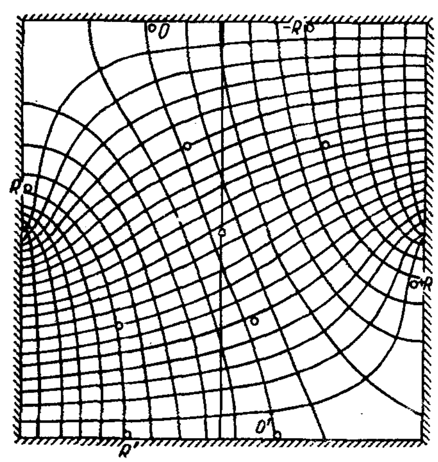

Fig. 4. Net of isotherms and flux lines constructed by means of an electro-hydrodynamical analog (the points lie on isotherms calculated by the method of successive conformal mappings).

Figure 4 shows the temperature and flux line field obtained by means of an electro-hydrodynamic analog for one of the cases discussed above. The results of analytic calculations are also shown for the RR' and 00' isotherms.

The quantity of heat calculated analytically is $Q = (\lambda/2\overline{K})\,\overline{K}'\,\Delta T$. The quantity of heat calculated from the isotherms constructed by means of the electro hydrodynamic analog, $Q = \Sigma\lambda\,(\Delta T/\Delta n)\,\Delta s$, varies between $Q_{min} = 0.63$ and $Q_{max} = 0.68$.

Clearly, this accuracy in the determination of the heat fluxes and in the construction of the isotherms is quite satisfactory in most technical applications.

NOTATION

λ – thermal conductivity; Q – heat flux; T – temperature; T – temperature difference between two adjacent isotherms; a and l – sides of a rectangle; Δs – element of an isotherm; n – element of a flux line.

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